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ON A CLASS OF OPTIMAL CONTROL PROBLEMS WITH AN ALMOST COST FREE SOLUTION

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On a class of optimal control problems with an almost cost free solution*)

by

J. Grasman

ABSTRACT

For a class of linear singular optimal control problems with a non-unique singular arc, the solution of the corresponding nearly singular problem is analyzed and a limit solution based on formal singular perturbations is derived. A rigourous proof of the correctness of the result is given by constructing a convergent power series satisfying the Riccati equation of the nearly singular problem.
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KEY WORDS & PHRASES: *Cheap control, singular perturbation*

*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

We consider the class of linear, time-invariant, n-dimensional dynamical systems

(1.1ab) $\dot{x} = Ax + Bv, \quad x(0) = x_0$

with performance index

(1.1c) $$J = \int_0^\infty x'Qx + \varepsilon^2 v'Rv \, dt, \quad 0 < \varepsilon << 1,$$

where $Q = C'C$ is a symmetric positive semi-definite matrix and R is symmetric and positive definite. We denote the n-dimensional state space by $\mathcal{X}$. The control vector takes its values in the linear n-dimensional space $\mathcal{U}$ and $v(\cdot): \mathbb{R}^+ \to \mathcal{U}$ is assumed to be a piece-wise continuous mapping. In this paper we analyze the problem of perfect regulation for a class of cheap optimal control problems of the type (1.1). For $\varepsilon = 0$ (1.1) reduces to a singular optimal control problem, which, as it is shown in [4], may have a family of solutions. As $\varepsilon \to 0$ the solution of (1.1) will tend to one of these solutions. In order to formulate such a class of singular problems in terms of A,B and Q we introduce some concepts of geometric system theory in section 2. For a more extensive exposition we refer to WONHAM [12]. In section 3 we specify the class of problems (1.1) to which our investigations apply and carry out some transformations in order to bring the system in its most suitable form. In section 4, a formal method for selecting the appropriate singular solution is presented, while in the sections 5 and 6 we prove the correctness of the result by developing the solution of (1.1) with respect to ε. It is remarked that the convergence of x satisfying (1.1) for $\varepsilon \to 0$ can also be proved by analyzing its Laplace transform see FRANCIS [2,3].

2. SOME CONCEPTS OF GEOMETRIC SYSTEM THEORY

Before giving a definition of controllabity subspaces we introduce the concept of (A,B)-invariant subspaces.

DEFINITION 2.1. A subspace $\mathcal{V} \subset \mathcal{X}$ is called (A,B)-invariant if for any $x_0 \in \mathcal{V}$ there exists a control $u(\cdot): \mathbb{R}^+ \to \mathcal{U}$ such that x(t) satisfying (1.1ab) remains in $\mathcal{V}$ for t > 0.

Let $\mathcal{B} = \mathrm{Im}B$. It can be proved that (A,B)-invariant subspaces may be characterized by the property $A\mathcal{V} \subset \mathcal{V} + \mathcal{B}$, or, equivalently, by the existence of a family of feedbacks

$$(2.1) \qquad \underline{F}(\mathcal{V}) = \{F: \mathcal{X} \to \mathcal{U} \rceil \ (A+BF)\ \mathcal{V} \subset \mathcal{V}\},$$

so that the closed loop system that starts $\mathcal{V}$ remains in $\mathcal{V}$ for t > 0. The class of (A,B)-invariant subspaces contained in some subspace of $\mathcal{X}$ is closed under addition and, thus, has a supremal element, see [12]. In the sequel we denote the supremal (A,B)-invariant subspace contained in $\mathcal{K} = \mathrm{Ker}Q$ by $\mathcal{V}_{\mathcal{K}}^*$.

DEFINITION 2.2. A subspace $\mathcal{R} \subset \mathcal{X}$ is called a controllability subspace if for any $x_0, x_1 \in \mathcal{R}$ there exists a T > 0 and a $u(\cdot): \mathbb{R}^+ \to \mathcal{U}$ such that x(t) given by (1.1ab) satisfies $x(T) = x_1$ and $x(t) \in \mathcal{R}$ for $0 < t < T$.

Clearly, a controllability subspace is also (A,B)-invariant. Given a subspace $\mathcal{B}_0 \subset \mathcal{X}$ and a mapping $A_F: \mathcal{X} \to \mathcal{X}$, we define the subspace $\mathcal{R}_0 \subset \mathcal{X}$ by

$$(2.2) \qquad \mathcal{R}_0 = \langle A_F \rceil \mathcal{B}_0 \rangle \equiv \mathcal{B}_0 + A_F\mathcal{B}_0 + \ldots + A_F^{n-1}\mathcal{B}_0.$$

It can be shown that $\mathcal{R}$ is a controllability subspace if and only if

$$(2.3) \qquad \mathcal{R} = \langle A+BF \rceil \mathcal{B} \cap \mathcal{R} \rangle \quad \text{for } F \in \underline{F}(\mathcal{R}).$$

Furthermore, the class of controllabity subspaces contained in some subspace of $\mathcal{X}$ is closed under addition and, thus, has a supremal element. The supremal controllability subspace contained in $\mathcal{K} = \mathrm{Ker}Q$ we denote by $\mathcal{R}_{\mathcal{K}}^*$. It can be proved that

$$(2.4) \qquad \mathcal{R}_{\mathcal{K}}^* = \langle A+BF \rceil \mathcal{B} \cap \mathcal{V}_{\mathcal{K}}^* \rangle \quad \text{for } F \in \underline{F}(\mathcal{V}_{\mathcal{K}}^*).$$

3. THE NEARLY SINGULAR OPTIMAL CONTROL PROBLEM

For the class of problems (1.1) we assume that

$$X = K + \mathcal{B}. \tag{3.1}$$

Furthermore, it is supposed that $\mathcal{R}_K^* \neq 0$ or $\mathcal{B} \cap K \neq 0$ as this property characterizes the class of problems we are aiming at, while condition (3.1) is meant as a restriction to focus our attention to a representive subclass for which the limit problem has a non-unique solution. In order to ensure finiteness and positive definiteness of the gain function we make the following hypotheses

(H3.1) the pair (A,B) is stabilizable,

(H3.2) the pair (C,A) is observable.

In addition to this similar type of hypotheses holding on a subsystem of (1.1) will be made later on.

The present study can be seen as the counterpart of the work by O'Malley and Jameson [8,9], where implicitly $\mathcal{R}_K^* = 0$. Since $AK \subset X = K + \mathcal{B}$, we have that $\mathcal{V}_K^* = K$ (see section 2). Let $\kappa = \dim K$. We assume that the state space X is the span of n orthonormal basis vectors $e_1,\dots,e_n$ chosen in such a way that K is the span of last κ of them. The control space $\mathcal{U}$ is the span of m orthonormal basis vectors $d_1,\dots,d_m$ chosen in such way that $B^{-1}e_1,\dots,B^{-1}e_{n-\kappa}$ has the same span as the first $n-\kappa$ basis vectors d_i. Here B^{-1} denotes the functional inverse of B, see [12, p.6]. By regular mappings $H: X \to X$ and $G: \mathcal{U} \to \mathcal{U}$ any system (A,B,Q,R) can be transformed into a system $(H^{-1}AH,\ H^{-1}BG,\ H'QH,\ G'RG)$ of the required form. Note that $H'QH$ and $G'RG$ are symmetric and positive (semi-)definite.

Consequently, we may restrict ourselves to systems (1.1) of the form

$$\begin{pmatrix}\dot{x}_s\\ \dot{x}_k\end{pmatrix} = \begin{pmatrix}A_s & A_{sk}\\ A_{ks} & A_k\end{pmatrix}\begin{pmatrix}x_s\\ x_k\end{pmatrix} + \begin{pmatrix}B_s & 0\\ 0 & B_k\end{pmatrix}\begin{pmatrix}v_s\\ v_k\end{pmatrix} \tag{3.2}$$

with $x_s = (x_1,\dots,x_{n-\kappa})$, $x_k = (x_{n-\kappa},\dots,x_n)$, $v_s = (v_1,\dots,v_{n-\kappa})$ and

$v_k = (v_{n-\kappa+1},\dots,v_m)$. For the control vector we write

$$(3.3)\qquad \begin{pmatrix} v_s \\ v_k \end{pmatrix} = \begin{pmatrix} 0 & -B_s^{-1}A_{sk} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_s \\ x_k \end{pmatrix} + \begin{pmatrix} u_s \\ u_k \end{pmatrix},$$

so that (1.1) becomes

$$(3.4ab)\qquad \begin{pmatrix} \dot{x}_s \\ \dot{x}_k \end{pmatrix} = \begin{pmatrix} A_s & 0 \\ A_{ks} & A_k \end{pmatrix}\begin{pmatrix} x_s \\ x_k \end{pmatrix} + \begin{pmatrix} B_s & 0 \\ 0 & B_k \end{pmatrix}\begin{pmatrix} u_s \\ u_k \end{pmatrix}, \begin{pmatrix} x_s(0) \\ x_k(0) \end{pmatrix} = \begin{pmatrix} x_{s0} \\ x_{k0} \end{pmatrix}$$

with performance index

$$(3.4c)\qquad J = \int_0^\infty (x_s',x_k')\begin{pmatrix} Q_s & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_s \\ x_k \end{pmatrix} + \varepsilon^2\Big[(x_s',x_k')\begin{pmatrix} 0 & 0 \\ 0 & M_k \end{pmatrix}\begin{pmatrix} x_s \\ x_k \end{pmatrix},$$

$$2(x_s',x_k')\begin{pmatrix} 0 & 0 \\ N_{ks} & N_k \end{pmatrix}\begin{pmatrix} u_s \\ u_k \end{pmatrix} + (u_s',u_k')\begin{pmatrix} R_s & R_{ks}' \\ R_{ks} & R_k \end{pmatrix}\begin{pmatrix} u_s \\ u_k \end{pmatrix}\Big]\, dt,$$

where $M_k = D_k'D_k = (B_s^{-1}A_{sk})'R_sB_s^{-1}A_{sk}$, $N_k = -(B_s^{-1}A_{sk})'R_{ks}'$ and $N_{ks} = (B_s^{-1}A_{sk})'R_s$. In the sequel we denote by A,B,Q,M,N and R the mappings of (3.4). We make the following additional hypotheses.

(H3.3) the pair (A_k,B_k) is stabilizable

(H3.4) the pair (D_k,A_k) is observable.

It is known that under the assumptions (H3.1) and (H3.2), (1.1), and thus (3.4), has an optimal solution with

$$(3.5)\qquad u = -\varepsilon^{-2}R^{-1}(B'P+\varepsilon^2N')x,$$

where P is a uniquely determined symmetric positive definite matrix satisfying the algebraic Riccati equation

$$(3.6)\qquad P(A-BR^{-1}N')+(A-BR^{-1}N')'P-\varepsilon^{-2}PBR^{-1}B'P+Q+\varepsilon^2(M-NR^{-1}N') = 0.$$

It is noted that in general this equation will have more than one solution.

The above statement is based on the work of WONHAM [11]. To be complete, we give this result in the following theorem.

THEOREM 3.1. *The class of symmetric, positive definite matrices contains a unique element that satisfies*

$$(3.7) \qquad A'P + PA - PBR^{-1}B'P + C'C = 0$$

provided that (A,B) *is stabilizable and* (C,A) *is observable.*

4. THE FORMAL LIMIT SOLUTION

Since the cost of control is small, it is expected that by some appropriately chosen initial pulse the solution will tend rapidly to the subspace K. In order to analyze this behaviour we carry out the following transformations

$$(4.1) \qquad u = \hat{u}/\varepsilon, \quad t = \tau\varepsilon \quad \text{and } J = \hat{J}\varepsilon.$$

Substituting (4.1) into (3.4) and formally letting $\varepsilon \to 0$ we obtain

$$(4.2a) \qquad d\hat{x}/d\tau = B\hat{u}$$

$$(4.2b) \qquad \hat{J} = \int_0^\infty \hat{x}'Q\hat{x} + \hat{u}'R\hat{u}\, d\tau.$$

We consider the feedback

$$(4.3a) \qquad \hat{u} = -R^{-1}B'\hat{P}\hat{x}$$

with $\hat{P}$ satisfying

$$(4.3b) \qquad \hat{P}BR^{-1}B'\hat{P} = Q.$$

Partitioning the inverse of R as

$$(4.4) \qquad R^{-1} = T = \begin{pmatrix} T_s & T'_{ks} \\ T_{ks} & T_k \end{pmatrix},$$

we write the unique positive semi-definite solution of (4.3b) as

$$(4.5a) \qquad P = \begin{pmatrix} P_{s0} & 0 \\ 0 & 0 \end{pmatrix}$$

with $P_{s0} > 0$ satisfying

$$(4.5b) \qquad P_{s0} B_s T_s B'_s P_{s0} = Q_s.$$

The corresponding closed loop system reads

$$(4.6) \qquad \begin{pmatrix} d\hat{x}_s/d\tau \\ d\hat{x}_k/d\tau \end{pmatrix} = \begin{pmatrix} -B_s T_s B'_s P_{s0} & 0 \\ -B_k T_{ks} B'_s P_{s0} & 0 \end{pmatrix} \begin{pmatrix} \hat{x}_s \\ \hat{x}_k \end{pmatrix}.$$

Integration yields

$$(4.7a) \qquad \hat{x}_s(\tau) = e^{-B_s T_s B'_s P_{s0} \tau} x_{s0},$$

$$(4.7b) \qquad \hat{x}_k(\tau) = x_{k0} - \int_0^\tau B_k T_{ks} B'_s P_{s0} \hat{x}_s(\bar{\tau}) d\bar{\tau}.$$

It is noted that $B_s T_s B'_s P_{s0} = P_{s0}^{-1} Q_s$ is positive definite. Consequently, as $\tau \to \infty$ $\hat{x}_s \to 0$ and $\hat{x}_k \to x_{k0} - \xi_{k0}$ with

$$(4.8) \qquad \xi_{k0} = B_k T_{ks} T_s^{-1} B_s^{-1} x_{s0}.$$

Letting $\varepsilon \to 0$, we observe that at the initial point the solution jumps from (x_{s0}, x_{k0}) to $(0, x_{k0} - \xi_{k0})$. Once the solution is in the subspace K it remains there as K is a invariant for (3.4). The performance index will be zero as $\varepsilon \to 0$ for any feedback $u_k = F_k x_k$. For the purpose of selecting the appropriate feedback we consider the optimal control problem for x_k given by (3.4ac) with $x_s = 0$ for $t > 0$:

$$(4.9ab) \qquad \dot{\bar{x}}_k = A_k \bar{x}_k + B_k \bar{u}_k, \quad \bar{x}_k(0) = x_{k0} - \xi_{k0}$$

(4.9c) $$\bar{J} = \int_0^\infty \bar{x}_k' M_k \bar{x}_k + 2\bar{x}_k' N_k \bar{u}_k + \bar{u}_k' R_k \bar{u}_k dt.$$

From (H3.2) and (H3.3) it follows that an optimal solution exists with

(4.10) $$\bar{u}_k = -R_k^{-1}(B_k'\bar{P}_k + N_k')\bar{x}_k,$$

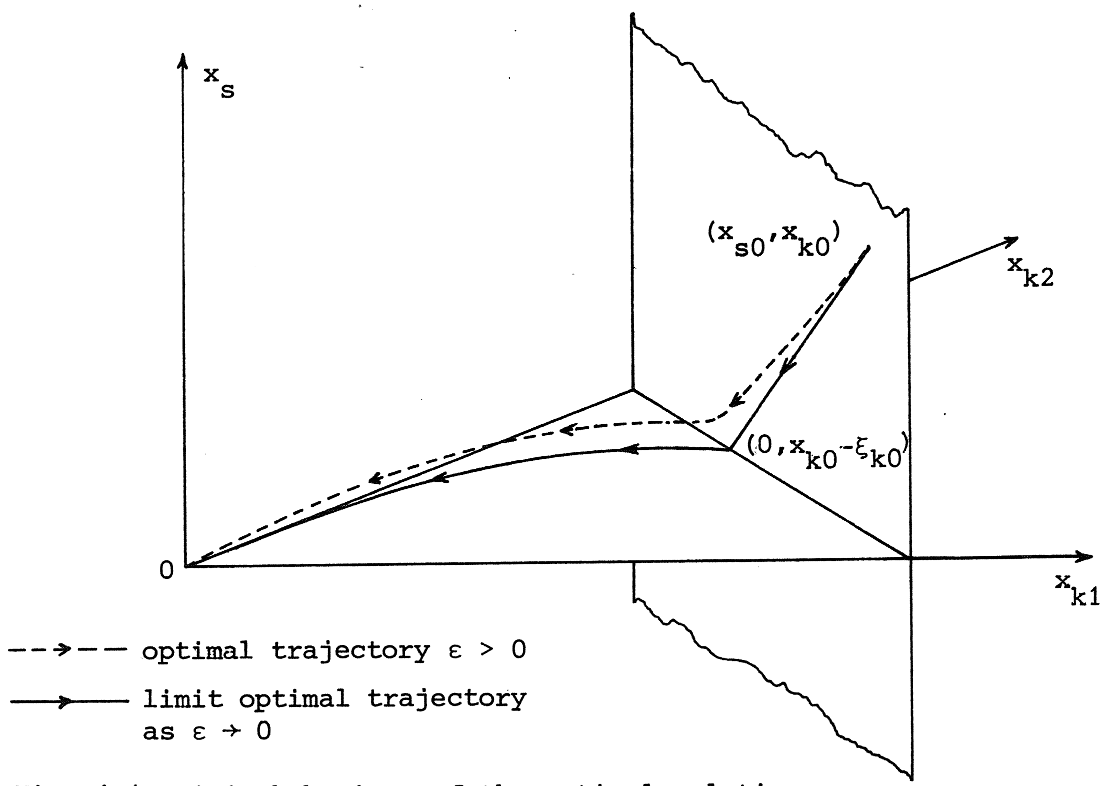
where $\bar{P}_k$ is the unique positive definite solution of the algebraic Riccati equation

(4.10b) $$\bar{P}_k(A_k - B_k R_k^{-1} N_k') + (A_k - B_k R_k^{-1} N_k')'\bar{P}_k - \bar{P}_k B_k R_k^{-1} B_k' \bar{P}_k + (M_k - N_k R_k^{-1} N_k') = 0,$$

see theorem 3.1. Thus, the optimal solution reads

(4.11) $$\bar{x}_k(t) = e^{(A_k - B_k R_k^{-1} B_k' \bar{P}_k - B_k R_k^{-1} N_k')t}(x_{k0} - \xi_{k0}).$$

<u>REMARK</u>. It is not obvious that $x_k(t,\varepsilon) \to \bar{x}_k(t)$ for $t \geq \delta > 0$ and $\varepsilon \to 0$, as $\bar{x}_k$ follows from the order $O(\varepsilon^2)$ terms of the performance index. Since $x_s = O(\varepsilon)$, x_s is also present with terms of order $O(\varepsilon^2)$, so before hand it is not clear that the system can be decomposed in the above way.



<u>Fig. 4.1</u>. Limit behaviour of the optimal solution

5. CONVERGENCE OF THE FORMAL POWER SERIES SATISFYING THE RICCATI EQUATION

The correctness of the formal result of section 4 will be proved in three steps. First we construct a formal power series satisfying (3.6). Then its convergence is proved by using a theorem of HAUTUS [5]. In this way we obtain a representation of an analytic solution of (3.6) for which we are able to prove positive definiteness. Finally, in section 6 we substitute this convergent series in the closed loop system and prove that the solution has the limit behaviour as stated in section 4.

Let us consider the formal power series

(5.1) $$P_\varepsilon = \varepsilon \sum_{j=0}^{\infty} P^{(j)}\varepsilon^j, \qquad P^{(j)} = \begin{pmatrix} P_{sj} & P'_{ksj} \\ P_{ksj} & P_{kj} \end{pmatrix}.$$

Susbtitution of (5.1) into (3.6) yields, by setting $\varepsilon = 0$, $P^{(0)} = \hat{P}$ with $\hat{P}$ given by (4.5). Equating the coefficients of the terms to ε we obtain

(5.2a) $$P_{s0}A_s + A'_sP_{s0} - P_{s1}B_sT_sB'_sP_{s0} - P_{s0}B_sT_sB'_sP_{s1} = 0$$

and

(5.2b) $$-P_{s0}B_sT_sN'_{ks} - P_{s0}B_sT_{sk}N'_k - P_{s0}B_sT_sB'_sP'_{ks1} - P_{s0}B_sT_{sk}B'_kP_{k1} = 0.$$

Consequently,

(5.3) $$P_{s1} = (B_sT_sB'_s)^{-1}A_s$$

and

(5.4) $$T_sN'_{ks} + T_{sk}N'_k + T_sB'_sP'_{ks1} + T_{sk}B'_kP_{k1} = 0.$$

Equating the terms of $0(\varepsilon^2)$ we obtain the equation

(5.5) $$-P_{ks1}B_sT_sN'_{ks} - P_{ks1}B_sT_{sk}N'_k + P_{k1}A_k - P_{k1}B_kT_{ks}N'_{ks} - P_{k1}B_kT_kN'_k$$
$$-N_{ks}T_sB'_sP_{ks1} - N_kT'_{ks}B'_sP'_{ks1} + A'_kP_{k1} - N_{ks}T'_{ks}B'_kP_{k1} - N_kT_kB'_kP_{k1}$$

$$-P_{ks1}B_sT_sB_s'P_{ks1}' - P_{ks1}B_sT_{sk}B_kP_{k1} - P_{k1}B_kT_{ks}B_s'P_{ks1}' - P_{k1}B_kT_kB_k'P_{k1}$$

$$+M_k - N_{ks}T_sN_{ks}' - N_{ks}T_{sk}N_k' - N_kT_{ks}N_{ks}' - N_kT_kN_k' = 0.$$

From (4.4) we derive that

$$R_k^{-1} = T_k - T_{ks}T_s^{-1}T_{ks}'. \tag{5.6}$$

Using (5.4) and (5.6) we reduce equation (5.5) to

$$P_{k1}[A_k - B_kR_k^{-1}N_k'] + [A_k - B_kR_k^{-1}N_k']'P_{k1} + \tag{5.7}$$

$$- P_{k1}B_kR_k^{-1}B_k'P_{k1} + M_k - N_kR_k^{-1}N_k' = 0,$$

which has a unique positive semi-definite solution $P_{k1} = \bar{P}_k$ see (4.10b). The equations for the higher order coefficients take the form

$$P_{sj}V_s + V_s'P_{sj} = F_s^{(j)} \tag{5.8a}$$

$$P_{kj}V_k + V_k'P_{kj} = F_k^{(j)} \tag{5.8b}$$

$$P_{ksj}V_s + P_{kj}V_{ks} = F_{ks}^{(j)} \tag{5.8c}$$

with

$$V_s = -B_sT_sB_s'P_{s0}, \quad V_{ks} = -B_kT_{ks}B_s'P_{s0}, \tag{5.9ab}$$

$$V_k = A_k - B_kR_k^{-1}N_k' - B_kR_k^{-1}B_k'P_{k1} \tag{5.9c}$$

and

$$F^{(j)} = F^{(j)}(P^{(0)},\ldots,P^{(j-1)}). \tag{5.10}$$

As V_s and V_k have eigenvalues with strictly negative real parts the above equations constitute an iteration process yielding uniquely determined coefficients $P^{(j)}$, $j = 2,3,\ldots$:

$$P_{pj} = \int_0^\infty e^{tV_p'} F_p^{(j)} e^{tV_p} dt, \quad p = s,k, \tag{5.11a}$$

$$P_{ksj} = -P_{kj} B_k T_{ks} T_s^{-1} B_s^{-1} + F_{ks}^{(j)}. \tag{5.11b}$$

In order to prove that (5.1) is the unique symmetric, positive definite solution of (3.6) we need the following lemma of HAUTUS [5, p.222] on the convergence of the formal power series (5.1). Furthermore, in a second lemma it is proved that (5.1) is positive definite indeed.

LEMMA 5.1. *Let the functions* $F_{ij}(Z;\varepsilon)$, $i,j = 1,\dots,n$ *with* $Z = \{Z_{ij}\}_{i,j=1,\dots,n}$ *be analytic in a neighbourhood* D *of* $(Z,\varepsilon) = (0,0)$ *and let* $Z = \hat{Z}(\varepsilon)$ *be a formal power series solution of the equation* $F(Z;\varepsilon) = 0$. *If* $\det F_Z(\hat{Z}(\varepsilon);\varepsilon) \neq 0$ *in a neighbourhood of* $\varepsilon = 0$ *then* $\hat{Z}(\varepsilon)$ *is convergent.*

LEMMA 5.2. *If the formal power series* (5.1) *is convergent with radius of convergence* ε_r, *then there exists an* ε_0 *such that for* $0 < \varepsilon < \varepsilon_0$ *this series is positive definite.*

PROOF. The matrix $P(\varepsilon)$ is positive definite if for any $x \neq 0$ the quadratic form $x'P(\varepsilon)x$ is positive. Since

$$x'P(\varepsilon)x = \varepsilon Q_1(x;\varepsilon) + O(\varepsilon^3|x|^2) \tag{5.12a}$$

with

$$Q_1(x;\varepsilon) = x_s'P_{s0}x_s + \varepsilon x_s'P_{s1}x_s + 2\varepsilon x_s'P_{ks1}'x_k + \varepsilon x_k'P_{k1}x_k, \tag{5.12b}$$

it suffices to prove that

$$Q_1(x;\varepsilon) > \delta\varepsilon|x|^2, \tag{5.13}$$

where δ is some arbitrary small positive number independent of ε. As P_{s0} and P_{k1} are positive definite we estimate the quadratic form as follows

$$Q_1(x;\varepsilon) > (\kappa^2+\varepsilon\eta)|x_s|^2 + \varepsilon\eta|x_k|^2 - 2\varepsilon M|x_s||x_k|, \tag{5.14}$$

where κ,η and $1/M$ are sufficiently small positive numbers. After subtracting a term $\varepsilon^2 M|x_s|^2/\kappa^2$ from the right-hand side of this inequality we obtain

$$(5.15) \qquad Q_1(x;\varepsilon) > (\kappa|x_s|-\varepsilon M|x_k|/\kappa)^2 - \varepsilon^2 M^2|x|^2/\kappa^2 + \varepsilon\eta|x|^2$$

so that inequality (5.13) holds for $\delta < \eta - \varepsilon M/\kappa^2$. Clearly, we have to choose $\varepsilon_0 = \min\{\varepsilon_r, \eta\kappa^2/M\}$. □

Now we are in the position to prove the main theorem of this section.

<u>THEOREM 5.1</u>. *The formal power series* (5.1) *has a positive radius of convergence and hence represents an analytic solution, defined in a neighbourhood of* $\varepsilon = 0$*, being the unique symmetric, positive definite solution of* (3.6).

<u>PROOF</u>. Substitution of

$$(5.16) \qquad P = \varepsilon P_0 + \varepsilon^2 P_1 + \varepsilon^2 Z$$

in (3.6) yields after multiplication of the equation with ε^{-1}

$$(5.17a) \qquad F(Z;\varepsilon) \equiv ZH(\varepsilon) + H(\varepsilon)'Z - \varepsilon ZBR^{-1}B'Z + \varepsilon G = 0$$

with

$$(5.17b) \qquad H(\varepsilon) = \varepsilon(A-BR^{-1}N') - BR^{-1}B'(P_0+\varepsilon P_1),$$

$$(5.17c) \qquad G = P_1(A-BR^{-1}N') + (A-BR^{-1}N')'P_1 - P_1BR^{-1}B'P_1 + (M-NR^{-1}N').$$

Using (5.3) and (5.4) we find that

$$H(\varepsilon) = \begin{pmatrix} H_s(\varepsilon) & 0 \\ H_{ks}(\varepsilon) & H_k(\varepsilon) \end{pmatrix}$$

with

$$H_s(\varepsilon) = -B_sT_sB_s'P_{s0} - \varepsilon B_sT_{ks}'B_{ks}'B_k'P_{ks1}, \qquad H_k(\varepsilon) = \varepsilon V_k,$$

$$H_{ks}(\varepsilon) = -B_k T_{ks} B_s' P_{s0} + \varepsilon A_{ks} - \varepsilon B_k T_{ks} B_s' P_{s1} - \varepsilon B_k T_k B_k' P_{ks1},$$

and V_k given by (5.9c). Consequently,

(5.8) $$\det(H) = \det(H_s)\det(H_k).$$

Because of hypotheses (H3.3) and (H3.4) the eigenvalues of H_k have strictly negative real parts and so $\det(H_k) \neq 0$. Since according to (4.5b)

$$B_s T_s B_s' P_{s0} = P_{s0}^{-1} Q_s$$

the determinant of H_s does not vanish either for ε sufficiently small, so that $\det(H) \neq 0$. This result is equivalent with $\det(\partial F/\partial Z) \neq 0$ at $Z = 0$. Then from lemma 5.1 and lemma 5.2 we conclude that (5.1) is convergent and positive definite. Hypotheses (H3.1) and (H3.2) guarantee that (3.6) has a unique symmetric, positive definite solution, see theorem 3.1. Since (5.1) converges, the series is a representation of this unique solution. □

6. THE SINGULARLY PERTURBED CLOSED LOOP SYSTEM

Substitution of (3.5) and (5.1) into (3.4ab) gives the closed loop system

(6.1ab) $$\begin{pmatrix}\dot{x}_s\\ \dot{x}_k\end{pmatrix} = \begin{pmatrix}\varepsilon^{-1}C_{ss}(\varepsilon) & C_{sk}(\varepsilon)\\ \varepsilon^{-1}C_{ks}(\varepsilon) & C_{kk}(\varepsilon)\end{pmatrix}\begin{pmatrix}x_s\\ x_k\end{pmatrix},\quad \begin{pmatrix}x_s(0)\\ x_k(0)\end{pmatrix} = \begin{pmatrix}x_{s0}\\ x_{k0}\end{pmatrix}$$

with

$$\varepsilon^{-1}C_{ss}(\varepsilon) = A_s - \varepsilon^{-2}B_s T_s B_s' P_s(\varepsilon) - \varepsilon^{-2}B_s T_{sk} B_k' P_{ks}(\varepsilon),$$

$$\varepsilon^{-1}C_{ks}(\varepsilon) = A_{ks} - \varepsilon^{-2}B_k T_{ks} B_s' P_s(\varepsilon) - \varepsilon^{-2}B_k T_k B_k' P_{ks}(\varepsilon),$$

$$C_{sk}(\varepsilon) = -\varepsilon^{-2}B_s T_s B_s' P_{ks}(\varepsilon) - B_s T_s N_{ks}' - \varepsilon^{-2}B_s T_{sk} B_k' P_k(\varepsilon) - B_s T_{sk} N_k'$$

$$C_{kk}(\varepsilon) = A_k - \varepsilon^{-2}B_k T_{ks} B_s' P_{ks}'(\varepsilon) - B_k T_{ks} N_{ks}' - \varepsilon^{-2}B_k T_k B_k' P_k(\varepsilon) - B_k T_k N_k'.$$

THEOREM 6.1. *Let* $(x_s(t), x_k(t))$ *be the solution of* (6.1ab), *then*

$$\left|\begin{pmatrix} x_s(t) \\ x_k(t) \end{pmatrix} - \begin{pmatrix} \hat{x}_s(t/\varepsilon) \\ \hat{x}_k(t/\varepsilon) \end{pmatrix} - \begin{pmatrix} 0 \\ \bar{x}_k(t) \end{pmatrix} + \begin{pmatrix} 0 \\ x_{k0} - \xi_{k0} \end{pmatrix}\right| = O(\varepsilon) \tag{6.2}$$

for $t > 0$ *with* $\hat{x}_s$, $\hat{x}_k$, $\bar{x}_k$ *and* ξ_{k0} *given by* (4.7)-(4.11).

PROOF. Since all eigenvalues of (6.1a) have negative real parts, see KWAKERNAAK and SIVAN [6, p.233], $|x_s|$ and $|x_k|$ have upperbounds of order $O(1)$. Integration of the equation for x_s yields

$$x_s(t) = e^{\varepsilon^{-1} C_{ss}(\varepsilon) t} x_{s0} + \int_0^t e^{\varepsilon^{-1} C_{ss}(\varepsilon)(t-\tau)} C_{sk}(\varepsilon) x_k(\tau) d\tau \tag{6.3}$$

or

$$x_s(t) = e^{\varepsilon^{-1} C_{ss}(0) t} x_{s0} + O(\varepsilon). \tag{6.4}$$

We now introduce the dependent variable

$$x_r = x_k - C_{ks}(\varepsilon) C_{ss}^{-1}(\varepsilon) x_s. \tag{6.5}$$

From (6.1a) we derive the corresponding differential equation

$$\dot{x}_r = [C_{kk}(\varepsilon) - C_{ks}(\varepsilon) - C_{ss}^{-1}(\varepsilon) C_{sk}(\varepsilon)]\{x_r + C_{ks}(\varepsilon) C_{ss}^{-1}(\varepsilon) x_s\}.$$

From (6.3) it follows that x_s is of the order $O(\varepsilon)$ in the L_1 norm, so that

$$x_r(t) = e^{[C_{kk}(\varepsilon) - C_{ks}(\varepsilon) C_{ss}^{-1}(\varepsilon) C_{sk}(\varepsilon)]t} \{x_{k0} - C_{ks}(\varepsilon) C_{ss}^{-1}(\varepsilon) x_{s0}\} + O(\varepsilon). \tag{6.6}$$

Substitution of (6.3) and (6.6) into (6.5) yields

$$x_k(t) = e^{[C_{kk}(0) - C_{ks}(0) C_{ss}^{-1}(0) C_{sk}(0)]t} \{x_{k0} - C_{ks}(0) C_{ss}^{-1}(0) x_{s0}\} + \tag{6.7}$$
$$- C_{ks}(0) C_{ss}^{-1}(0) e^{\varepsilon^{-1} C_{ss}(0) t} x_{s0} + O(\varepsilon).$$

It is noted that

(6.8) $$C_{ks}(0)C_{ss}^{-1}(0) = B_k T_{ks} T_s^{-1} B_s^{-1},$$

so that

(6.9) $$C_{kk}(0) - C_{ks}(0)C_{ss}^{-1}(0)C_{sk}(0) = A_k - B_k R_k^{-1} B_k' \bar{P}_k - B_k R_k^{-1} N_k'.$$

According to (6.4) and (6.7) x_s and x_k satisfy (6.2), which completes the proof. □

Thus, we have found that the expression

(6.10) $$\begin{pmatrix} \tilde{x}_s(t;\varepsilon) \\ \tilde{x}_k(t;\varepsilon) \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{x}_k(t) \end{pmatrix} + \begin{pmatrix} \hat{x}_s(t/\varepsilon) \\ \hat{x}_k(t/\varepsilon) \end{pmatrix} - \begin{pmatrix} 0 \\ x_{k0} - \xi_{k0} \end{pmatrix}$$

is a uniformly valid asymptotic approximation up to $O(\varepsilon)$ of the optimal solution of (3.4) for $\varepsilon > 0$.

7. AN EXAMPLE

As an illustration of the method of approximating the solution of a nearly singular system we present the following example

(7.1a) $$\dot{x} = Ax + Bv, \quad x(0) = x_0,$$

(7.1b) $$J = \int_0^\infty x'Qx + \varepsilon^2 v'Rv \, dt$$

with

(7.1c) $$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Putting (7.1) in the required form (3.5) we obtain

(7.2a) $$\dot{x}_1 = u_1, \quad x_1(0) = x_{10},$$

(7.2b) $\dot{x}_2 = x_1+u_2, \quad x_2(0) = x_{20},$

(7.3c) $$J = \int_0^\infty x_1^2 + \varepsilon^2(x_2^2-2x_2u_1+u_1^2+u_2^2)\,dt.$$

In the limit $\varepsilon \to 0$ the system jumps initially from (x_{10},x_{20}) to $(0,x_{20})$, see (4.7) and (4.8). In order to analyze the limit solution in the subspace $x_1 = 0$ for $t > 0$, we consider the optimal control problem (4.9) for the system (7.2), so

(7.4a) $\dot{\bar{x}}_2 = \bar{u}_2, \quad \bar{x}_2(0) = x_{20},$

(7.4b) $$J = \int_0^\infty \bar{x}_2^2 + \bar{u}_2^2\,dt.$$

The optimal solution satisfies $\bar{u}_2 = -\bar{x}_2$, see (4.10).

We are able to verify the result as the exact solution of the problem (7.1) can be computed; the algebraic Riccati equation reads

(7.5) $$Q + P_\varepsilon A' + A'P_\varepsilon - \varepsilon^{-2}P_\varepsilon BR^{-1}B'P_\varepsilon = 0,$$

which has the positive definite solution

(7.6) $$P_\varepsilon = \begin{pmatrix} \varepsilon\sqrt{1+\varepsilon^2} & \varepsilon^2 \\ \varepsilon^2 & \varepsilon^2 \end{pmatrix}.$$

Since $u_\varepsilon = -\varepsilon^{-2}R^{-1}B'P_\varepsilon x_\varepsilon$, the closed loop system reads

(7.7a) $\dot{x}_{\varepsilon 1} = -\varepsilon^{-1}\sqrt{1+\varepsilon^2}\; x_{\varepsilon 1},$

(7.7b) $\dot{x}_{\varepsilon 2} = -x_{\varepsilon 2},$

Consequently the exact solution converges to the given limit solution as $\varepsilon \to 0$.

8. CONCLUDING REMARKS

The main result of this paper concerns the limit behaviour of solutions of nearly singular optimal control problems (1.1) with a nontrivial controllability subspace contained in the kernel of Q. For this class of problems the corresponding simgular optimal control problems, that is (1.1) with $\varepsilon = 0$, has a family of solutions, each of them composed of an initial pulse and a singular arc. In section 4 we presented a method for selecting the correct limit solution. Since this procedure is far from straightforward, a rigourous proof of the correctness of the result is in its place. It is remarked that this analysis based on a theorem by HAUTUS [5] also applies to singular perturbation solutions of problems, for which the singular optimal control problem has a unique solution, see [8] and [9]. Then it suffices to compute the first order asymptotic approximation of the positive definite solution of the Riccati equation.

The present study was motivated by Wonham's geometric approach of multivariate systems. Analyzing cheap control problems in this way we arrived at the problem formulated in [4]. It is along these lines that we expect that the concept of almost invariant subspaces, introduced by WILLEMS [10], may reveal new geometrical aspects of cheap control, which in turn can be of great help if one tries to solve other problems in this field. As open problems under investigation we mention the class (1.1) with condition (3.1) replaced by

$$\mathcal{X} = \mathcal{V}_{\mathcal{K}}^{*} + \mathcal{B} + A\mathcal{B} + \ldots + A^{k}\mathcal{B},$$

or by

$$\mathcal{X} \supset \mathcal{K} + \mathcal{B}.$$

In the last case the optimal solution is not necessarily almost cost free.

The class of control systems we analyzed has various applications in technology. As an example we mention the use in solar energy, see [1]. Actually in a study of OBERLE [7, p.302], a model of optimal heating and cooling to solar energy is proposed, which belongs to the class of singular optimal controls with a nonunique singular arc. This fact may for a great deal explain the observed sensitivity of the singular arc on the physical parameters.

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[1] DORATO, P., *A review of the application of modern control theory to solar energy systems*. Proceedings 18th IEEE conference on decision and control, Fort Lauderdale, Florida (1979), p.907-910.

[2] FRANCIS, B.A., *On totally singular linear-quadratic optimal control,* Report R-77-7 Dept. Electr. Engin. McGill Univ. (1977).

[3] FRANCIS, B.A., *Singularly perturbed linear initial value problems with an application to singular optimal controls,* (preprint).

[4] GRASMAN, J., *Non-uniqueness in singular optimal control,* Int. Symp. on Math. Theory of Networks and Systems vol. 3 (1979), p.415-420.

[5] HAUTUS, M.L.J., *Formal on convergent solutions of ordinary differential equations,* Proc. Kon. Ned. Akad. Wetensch., Amsterdam, Series A 81 (1978), 216-229.

[6] KWAKERNAAK, H. & R. SIVAN, *Linear optimal control systems,* Wiley-Interscience, New York (1972).

[7] OBERLE, H.J., *Numerical computation of singular control problems with application to optimal heating and cooling by solar energy,* Appl. Math. Optim. 5 (1979), p.297-314.

[8] O'MALLEY, R.E. & A. JAMESON, *Singular perturbations and singular arcs,* part I, IEEE Trans. on Aut. Control, vol. AC-20 (1975), p.218-226.

[9] O'MALLEY, R.E. & A. JAMESON, *Singular perturbations and singular arcs,* part II, IEEE Trans. on Aut. Control, vol. AC-22 (1977), p.328-337.

[10] WILLEMS, J.C, *Almost invariant subspaces: an approach to high gain feedback design,* Part I: *Almost controlled invariant subspaces,* Submitted to IEEE Transactions on Automatic Control.

[11] WONHAM, W.M., *On a matrix Riccati equation of stochastic control,* SIAM J. Control 6 (1968), p.681-697.

[12] WONHAM, W.M., *Linear multivariable control: A geometric approach,* Lecture Notes in Econ. and Math. Systems vol. 101 (1974).